Chaotic behavior of disordered Hamiltonian systems

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Outline

- **Disordered 1D lattices:**
  - ✓ **The quartic disordered Klein-Gordon (DKG) model**
  - ✓ **The disordered discrete nonlinear Schrödinger equation (DDNLS)**
  - ✓ **Different dynamical behaviors**
- **Chaotic behavior of the DKG and DDNLS models**
  - ✓ **Lyapunov exponents**
  - ✓ **Deviation Vector Distributions**
- **Summary**

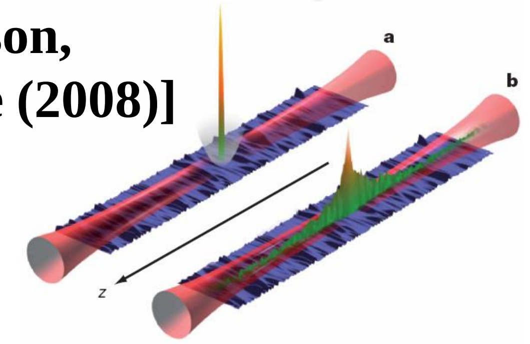
Work in collaboration with

Bob Senyange (PhD student): DKG model

Bertin Many Manda (PhD student): DDNLS model

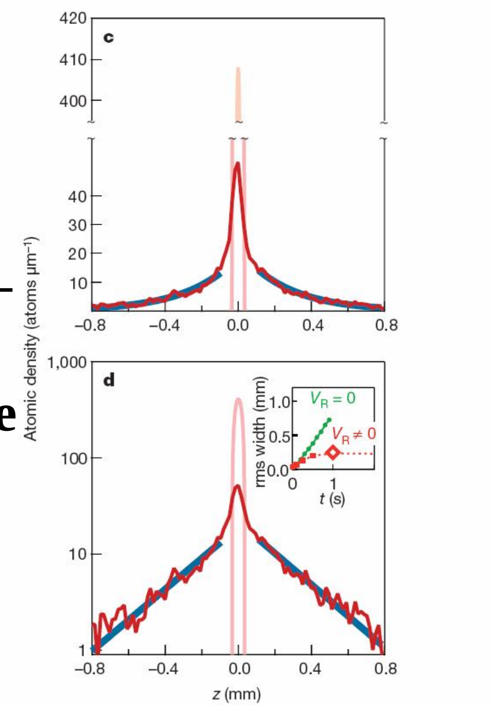
Interplay of disorder and nonlinearity

Waves in disordered media – Anderson localization [Anderson, Phys. Rev. (1958)]. Experiments on BEC [Billy et al., Nature (2008)]

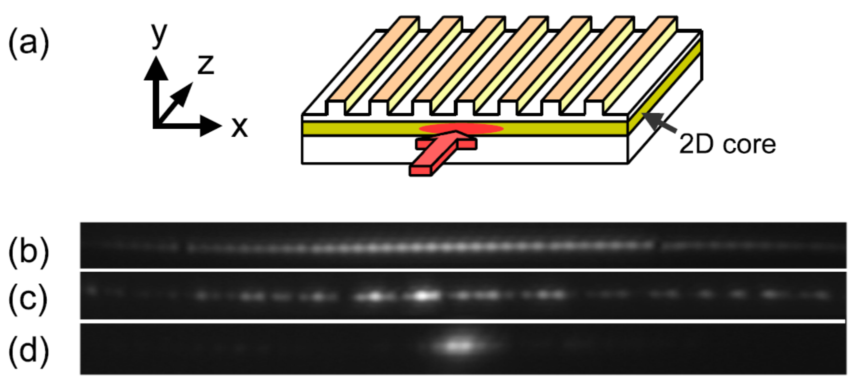


Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL (1993) – Molina, Phys. Rev. B (1998) – Pikovsky & Shepelyansky, PRL (2008) – Kopidakis et al., PRL (2008) – Flach et al., PRL (2009) – S. et al., PRE (2009) – Mulansky & Pikovsky, EPL (2010) – S. & Flach, PRE (2010) – Laptyeva et al., EPL (2010) – Mulansky et al., PRE & J.Stat.Phys. (2011) – Bodyfelt et al., PRE (2011) – Bodyfelt et al., IJBC (2011)]



Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)]



The disordered Klein – Gordon (DKG) model

$$H_K = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W}\left(u_{l+1} - u_l\right)^2$$

with fixed boundary conditions $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically N=1000.

Parameters: W and the total energy E. $\tilde{\varepsilon}_l$ chosen uniformly from $\left[\frac{1}{2}, \frac{3}{2}\right]$.

Linear case (neglecting the term $u_l^4/4$)

Ansatz: $u_l=A_l \exp(i\omega t)$. Normal modes (NMs) $A_{\nu,l}$ - Eigenvalue problem:

$$\lambda A_l = \varepsilon_l A_l - (A_{l+1} + A_{l-1}) \text{ with } \lambda = W\omega^2 - W - 2, \quad \varepsilon_l = W(\tilde{\varepsilon}_l - 1)$$

The disordered discrete nonlinear Schrödinger (DDNLS) equation

We also consider the system:

$$H_D = \sum_{l=1}^{N} \varepsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right)$$

where ε_l chosen uniformly from $\left[-\frac{W}{2}, \frac{W}{2}\right]$ and β is the nonlinear parameter.

Conserved quantities: The energy and the norm $S = \sum_l |\psi_l|^2$ of the wave packet.

Distribution characterization

We consider normalized energy distributions $z_\nu \equiv \frac{E_\nu}{\sum_m E_m}$

with $E_\nu = \frac{p_\nu^2}{2} + \frac{\tilde{\varepsilon}_\nu}{2} u_\nu^2 + \frac{1}{4} u_\nu^4 + \frac{1}{4W} \left(u_{\nu+1} - u_\nu \right)^2$ for the DKG model,

and norm distributions $z_\nu \equiv \frac{|\psi_\nu|^2}{\sum_l |\psi_l|^2}$ for the DDNLS system.

Second moment: $m_2 = \sum_{\nu=1}^{N} \left(\nu - \bar{\nu} \right)^2 z_\nu$ with $\bar{\nu} = \sum_{\nu=1}^{N} \nu z_\nu$

Participation number: $P = \frac{1}{\sum_{\nu=1}^{N} z_\nu^2}$

measures the number of stronger excited modes in z_ν.

Single site $P=1$. Equipartition of energy $P=N$.

Scales

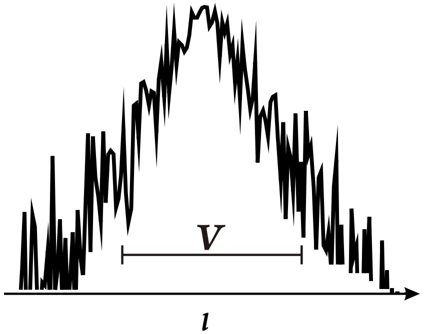
Linear case: $\omega_\nu^2 \in \left[\frac{1}{2}, \frac{3}{2} + \frac{4}{W}\right]$**, width of the squared frequency spectrum:**

$$\Delta_K = 1 + \frac{4}{W}$$

$$(\Delta_D = W + 4)$$

Localization volume of an eigenstate:

$$V \sim \frac{1}{\sum_{l=1}^{N} A_{\nu,l}^4}$$



Average spacing of squared eigenfrequencies of NMs within the range of a localization volume: $d_K \approx \frac{\Delta_K}{V}$

Nonlinearity induced squared frequency shift of a single site oscillator

$$\delta_l = \frac{3E_l}{2\tilde{\varepsilon}_l} \propto E \qquad (\delta_l = \beta |\psi_l|^2)$$

The relation of the two scales $d_K \leq \Delta_K$ with the nonlinear frequency shift δ_l determines the packet evolution.

Different Dynamical Regimes

Three expected evolution regimes **[Flach, Chem. Phys (2010) - S. & Flach, PRE (2010) - Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)]**

Δ: width of the frequency spectrum, d: average spacing of interacting modes,

δ: nonlinear frequency shift.

Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky, & Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \rightarrow m_2 \sim t^{1/3}$

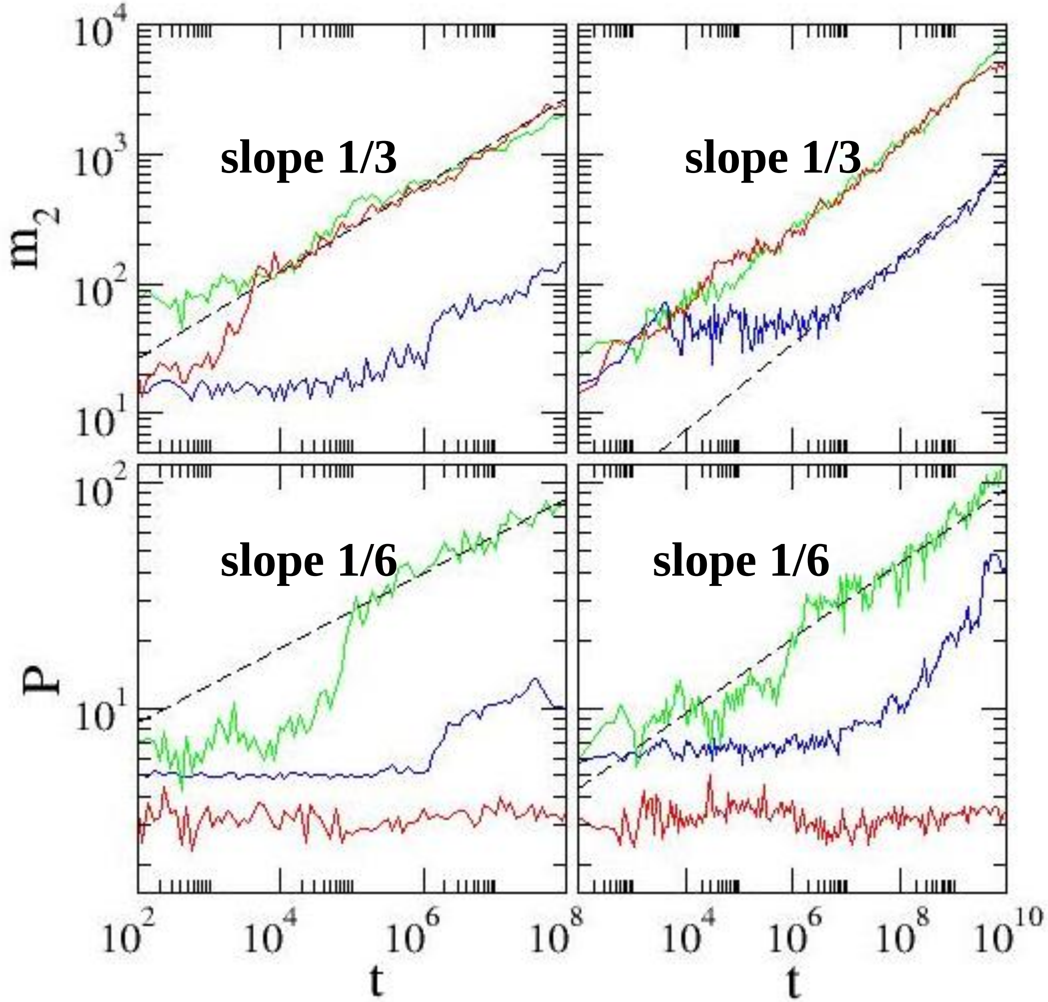
Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: $\delta>\Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Single site excitations

DDNLS W=4, β= 0.1, 1, 4.5 **DKG** W = 4, E = 0.05, 0.4, 1.5



No strong chaos regime

In weak chaos regime we averaged the measured exponent α (m_2~t^α) over 20 realizations:

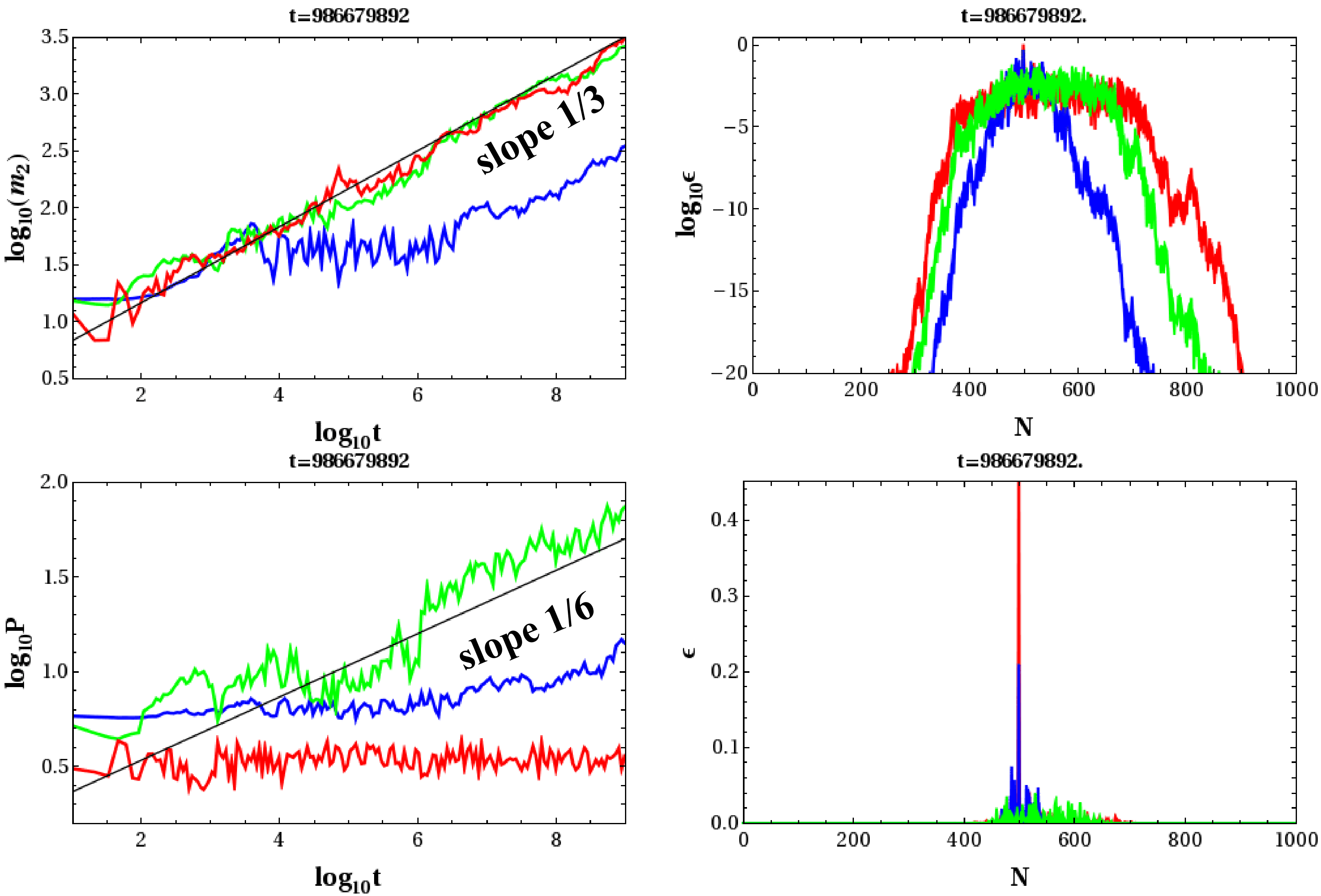
α=0.33±0.05 (DKG)

α=0.33±0.02 (DDLNS)

Flach et al., PRL (2009)

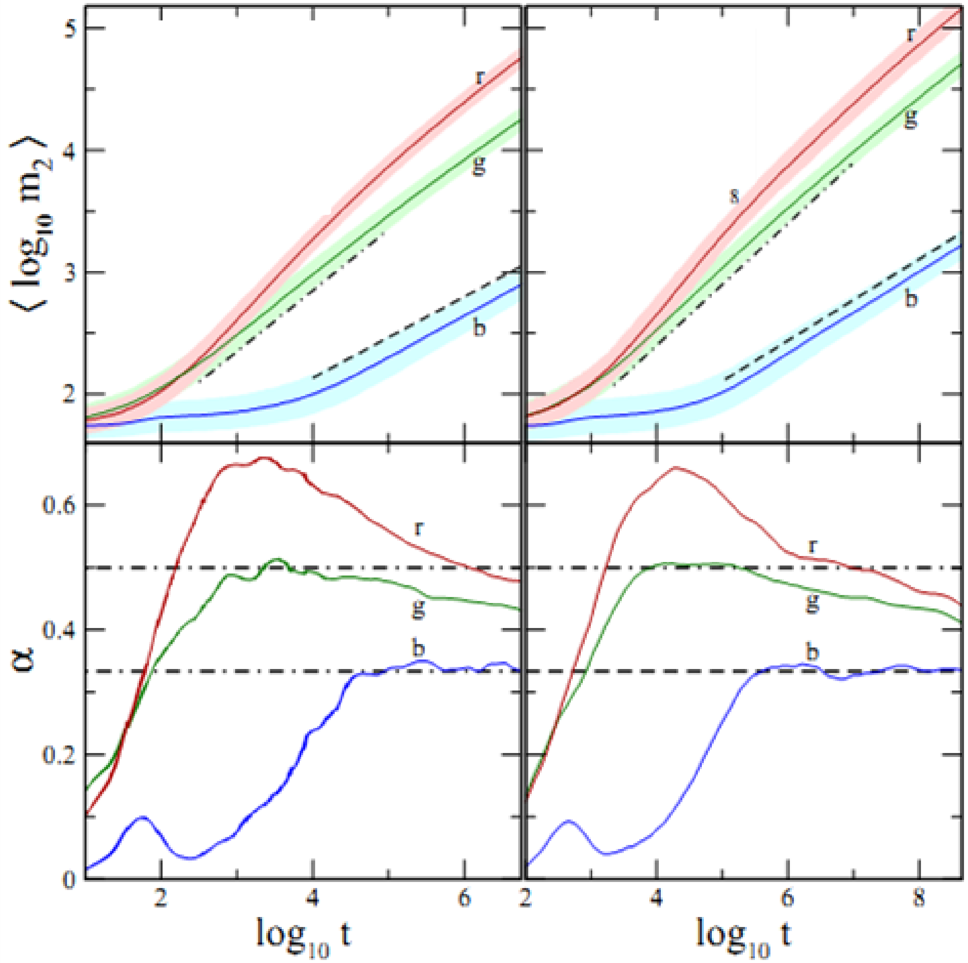
S. et al., PRE (2009)

DKG: Different spreading regimes



Crossover from strong to weak chaos (block excitations)

DDNLS β= 0.04, 0.72, 3.6 DKG E= 0.01, 0.2, 0.75



W=4

Average over 1000 realizations!

$$\alpha(\log t) = \frac{d\langle \log m_2 \rangle}{d \log t}$$

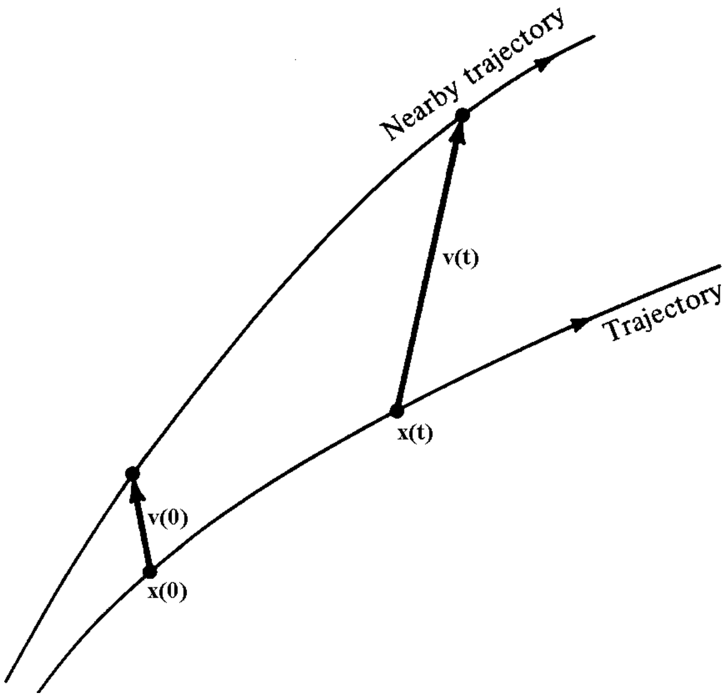
Laptyeva et al., EPL (2010)

Bodyfelt et al., PRE (2011)

Variational Equations

We use the notation $\mathbf{x} = (q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N)^T$. The **deviation vector** from a given orbit is denoted by

$$\mathbf{v} = (\delta x_1, \delta x_2,\ldots,\delta x_n)^T, \text{ with } n=2N$$



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

The time evolution of **v** is given by the so-called **variational equations:**

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}, \quad P_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \quad i,j = 1,2,\ldots,n$$

Maximum Lyapunov Exponent

Chaos: sensitive dependence on initial conditions.

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\mathrm{mLCE} = \lambda_1 = \lim_{t\to\infty} \Lambda(t) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

$\lambda_1 = 0 \rightarrow$ Regular motion

$\lambda_1 \neq 0 \rightarrow$ Chaotic motion

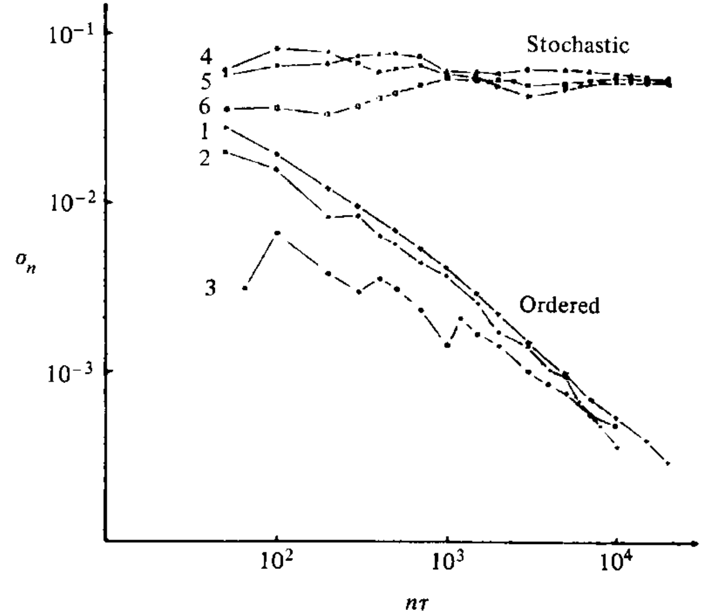


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

Symplectic integration

We apply the 2-part splitting integrator ABA864 [Blanes et al., Appl. Num. Math. (2013) – Senyange & S., EPJ ST (2018)] to the DKG model:

$$H_K = \sum_{l=1}^{N} \left(\frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} \left(u_{l+1} - u_l \right)^2 \right)$$

and the 3-part splitting integrator $ABC^6_{[SS]}$ [S. et al., Phys. Let. A (2014) – Gerlach et al., EPJ ST (2016) – Danieli et al., MinE (2019)] to the DDNLS system:

$$H_D = \sum_l \varepsilon_l \left|\psi_l\right|^2 + \frac{\beta}{2} \left|\psi_l\right|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l \right), \qquad \psi_l = \frac{1}{\sqrt{2}} \left(q_l + i p_l \right)$$

$$H_D = \sum_l \left(\frac{\varepsilon_l}{2} \left(q_l^2 + p_l^2 \right) + \frac{\beta}{8} \left(q_l^2 + p_l^2 \right)^2 - q_n q_{n+1} - p_n p_{n+1} \right)$$

By using the so-called Tangent Map method we extend these symplectic integration schemes in order to integrate simultaneously the variational equations [S. & Gerlach, PRE (2010) – Gerlach & S., Discr. Cont. Dyn. Sys. (2011) – Gerlach et al., IJBC (2012)].

Symplectic integration

For more information on the various symplectic integrators we can use for integrating

- **the equations of motion, and**
- **the variational equations**

of multidimensional Hamiltonian systems see Poster 14:

Efficient integration schemes for multidimensional disordered nonlinear Hamiltonian lattices

Bob Senyange[1,2] and Haris Skokos[1]

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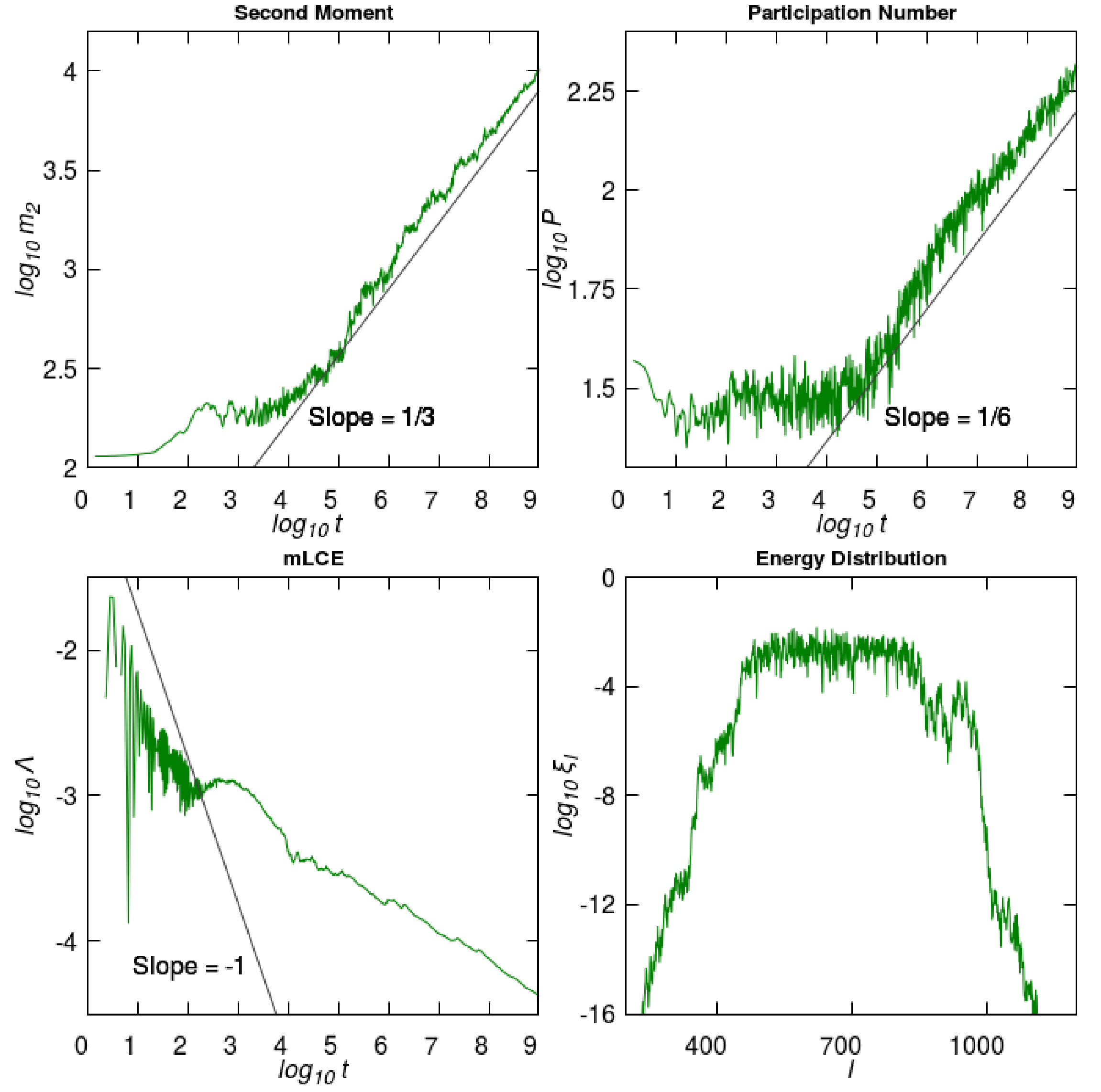
2. Department of Mathematics, Muni University, Arua 725, Uganda.

by Bob Senyange

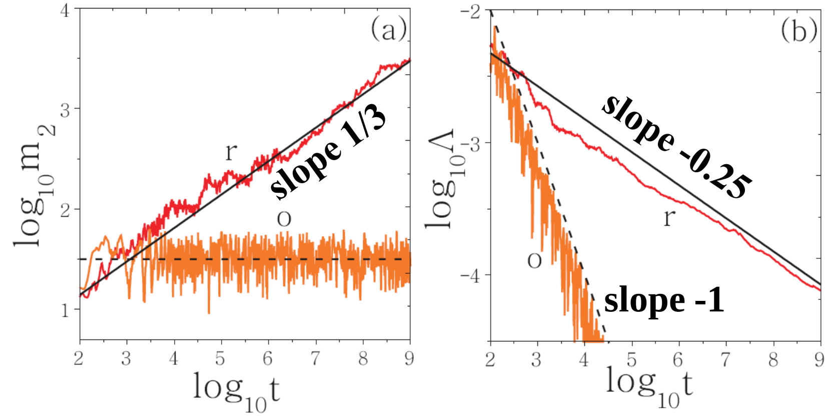
DKG: Weak Chaos

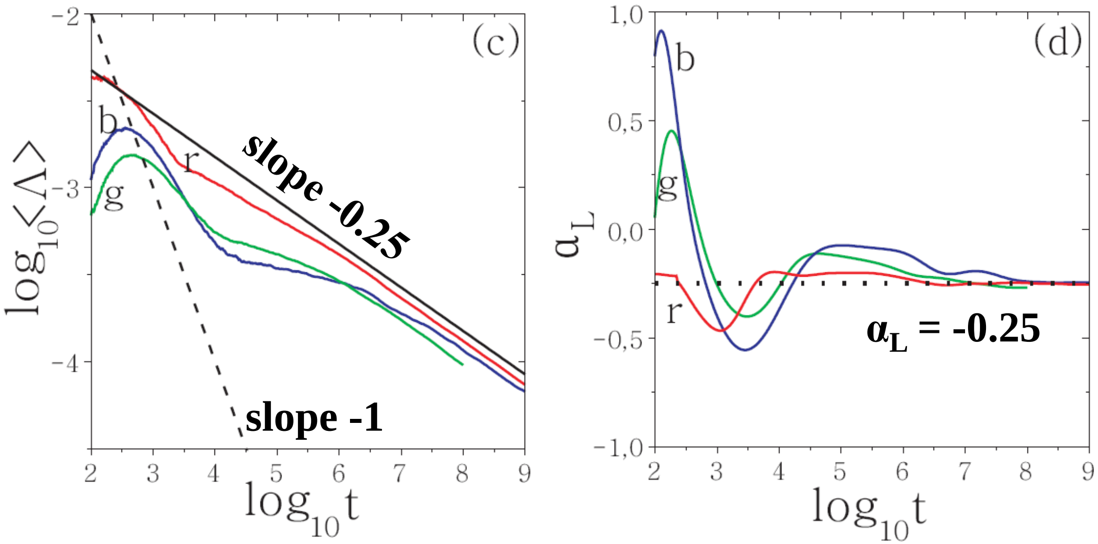


Block excitation
L=37 sites,
E=0.37, W=3

DKG: Weak Chaos

Individual runs
Linear case
E=0.4, W=4





$$\alpha_L = \frac{d\left(\log\langle\Lambda\rangle\right)}{d\log t}$$

Average over 50 realizations

Single site excitation E=0.4, W=4

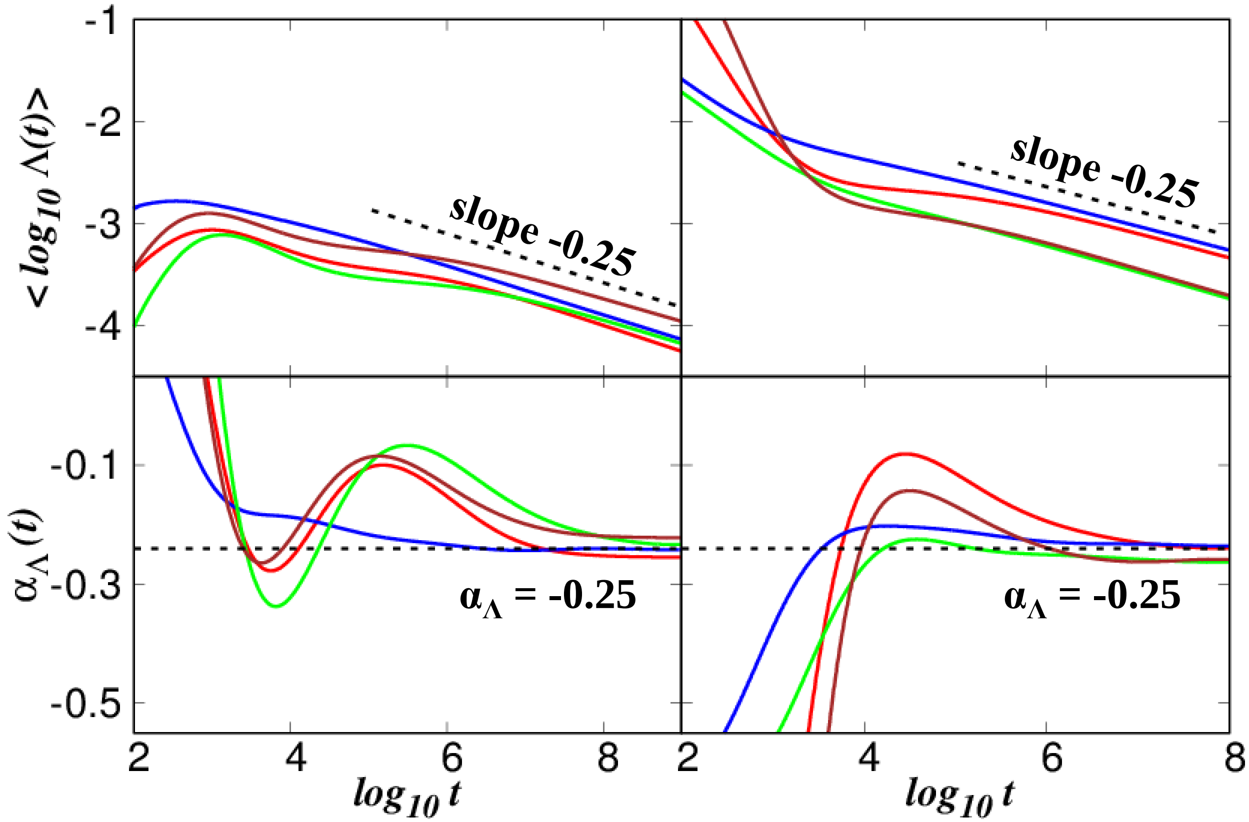
Block excitation (L=21 sites) E=0.21, W=4

Block excitation (L=37 sites) E=0.37, W=3

S. et al., PRL (2013)

Weak Chaos: DKG and DDNLS

DKG



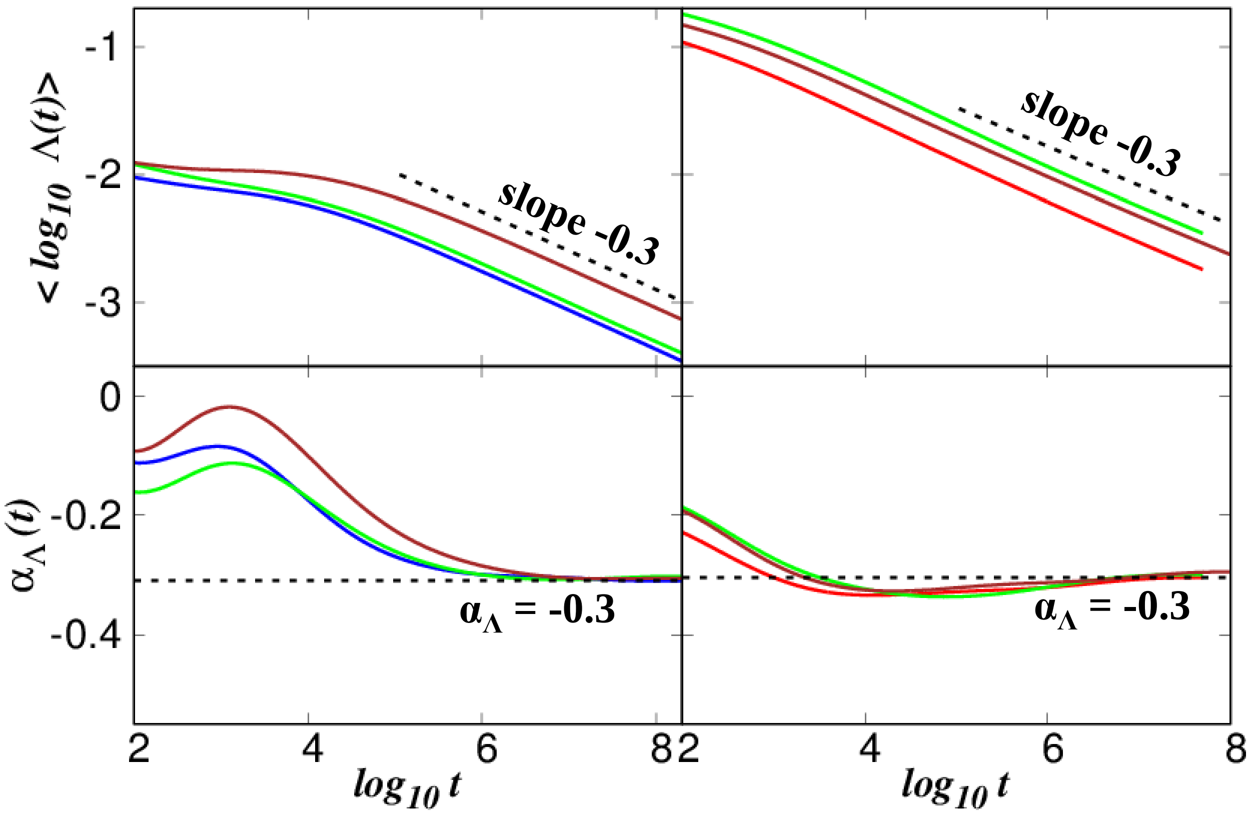
DDNLS

Average over 100 realizations [Senyange, Many Manda & S., PRE (2018)]

Block excitation (L=37 sites) E=0.37, W=3	**Block excitation (L=21 sites) β=0.04, W=4**
Single site excitation E=0.4, W=4	**Single site excitation β=1, W=4**
Block excitation (L=21 sites) E=0.21, W=4	**Single site excitation β=0.6, W=3**
Block excitation (L=13 sites) E=0.26, W=5	**Block excitation (L=21 sites) β=0.03, W=3**

Strong Chaos: DKG and DDNLS

DKG



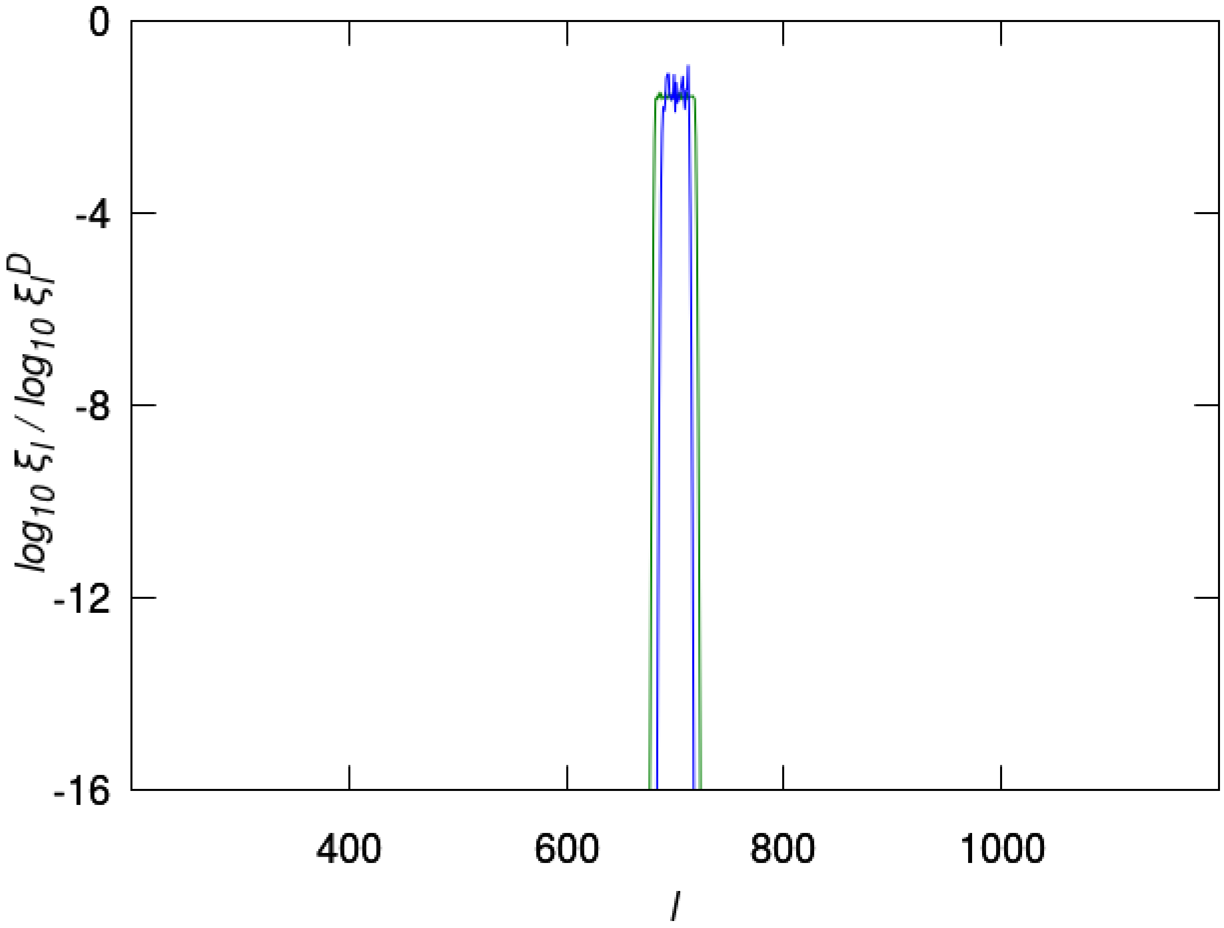
DDNLS

Average over 100 realizations [Senyange, Many Manda & S., PRE (2018)]

Block excitation (L=83 sites) E=0.83, W=2
Block excitation (L=37 sites) E=0.37, W=3
Block excitation (L=83 sites) E=0.83, W=3

Block excitation (L=21 sites) β=0.62, W=3.5
Block excitation (L=21 sites) β=0.5, W=3
Block excitation (L=21 sites) β=0.72, W=3.5

Deviation Vector Distributions (DVDs)



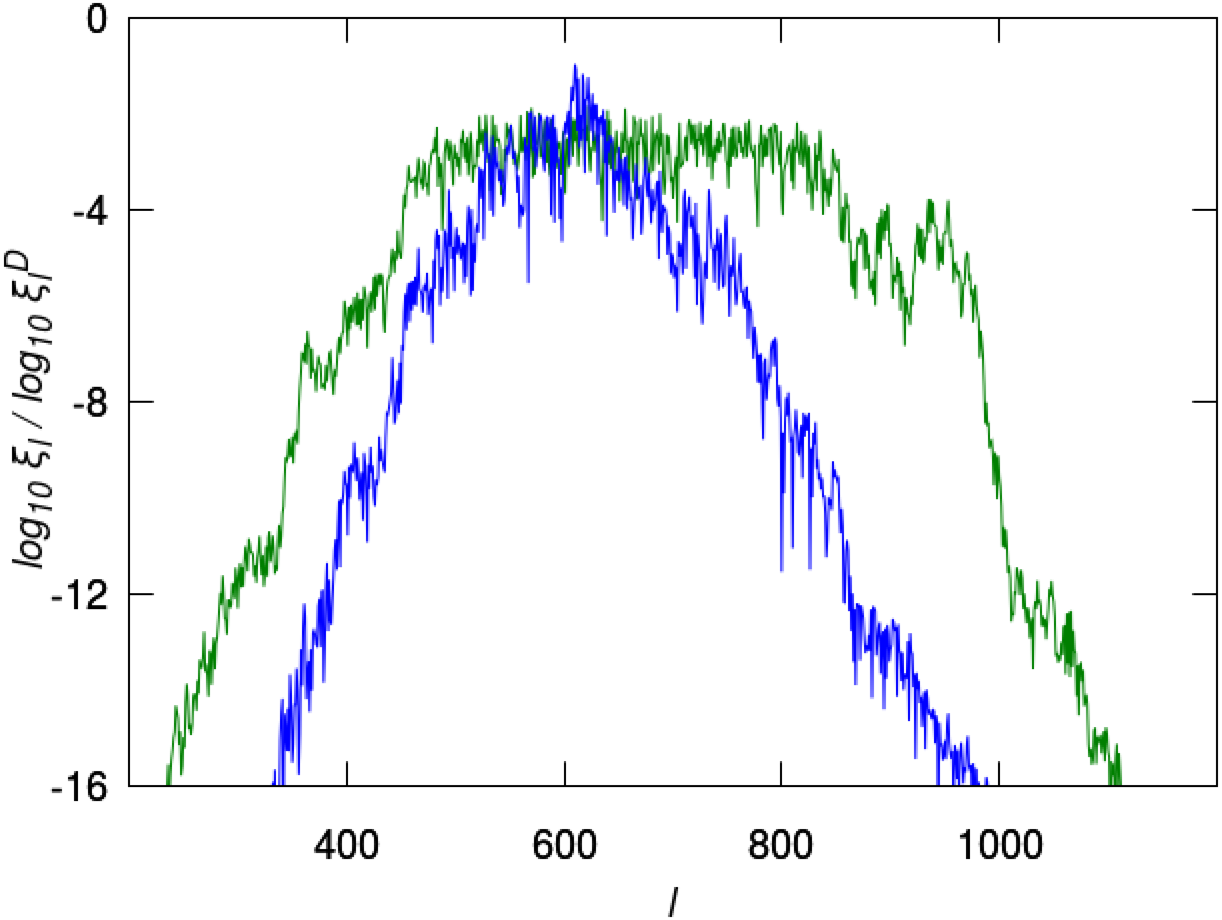
Energy
DVD

DKG
weak chaos
L=37 sites,
E=0.37, W=3

Deviation vector:
v(t)=(δu_1(t), δu_2(t),…, δu_N(t), δp_1(t), δp_2(t),…, δp_N(t))

DVD: $$\xi_l^D = \frac{\delta u_l^2 + \delta p_l^2}{\sum_l \left(\delta u_l^2 + \delta p_l^2\right)}$$

Deviation Vector Distributions (DVDs)



Energy
DVD

DKG
weak chaos
L=37 sites,
E=0.37, W=3

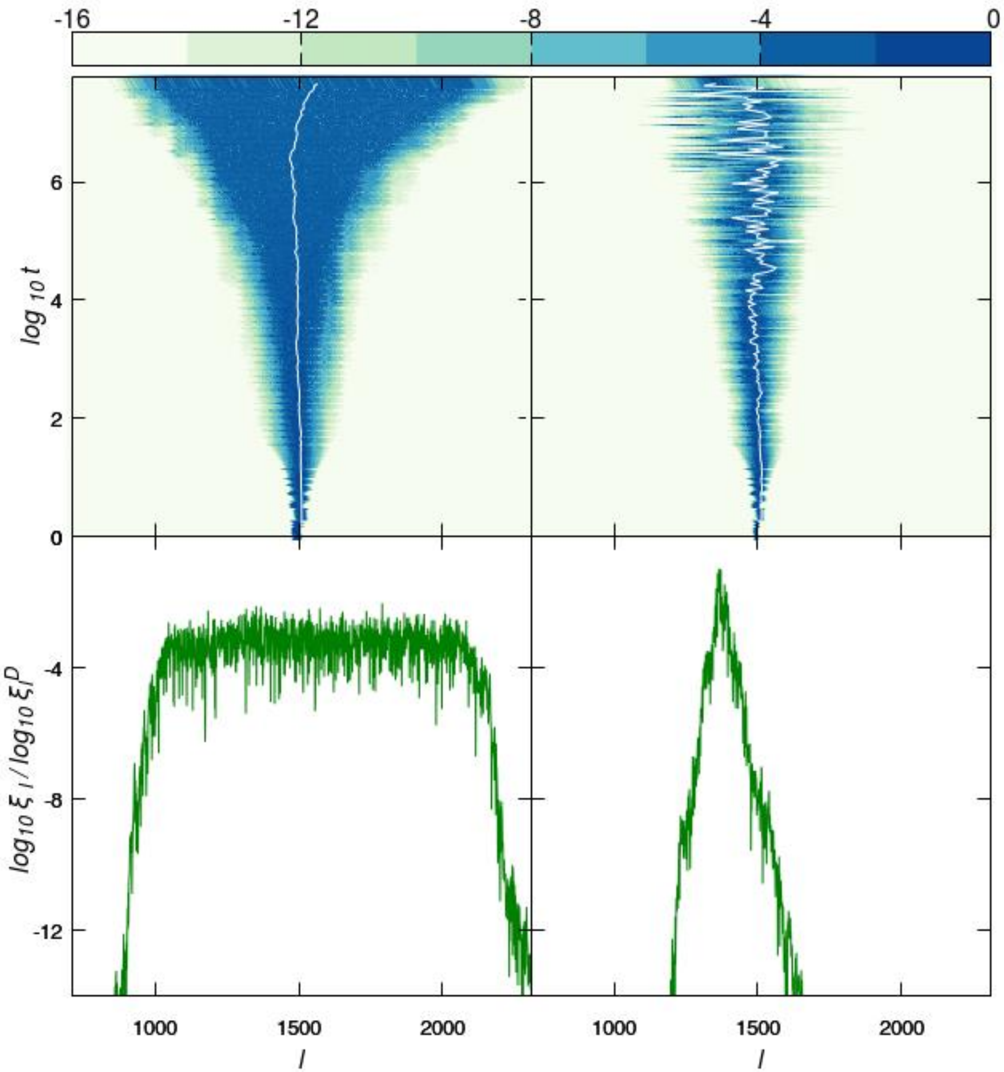
Deviation vector:
$v(t)=(\delta u_1(t), \delta u_2(t),\dots, \delta u_N(t), \delta p_1(t), \delta p_2(t),\dots, \delta p_N(t))$

DVD: $\xi_l^D = \dfrac{\delta u_l^2 + \delta p_l^2}{\sum_l \left(\delta u_l^2 + \delta p_l^2\right)}$

Deviation Vector Distributions (DVDs)

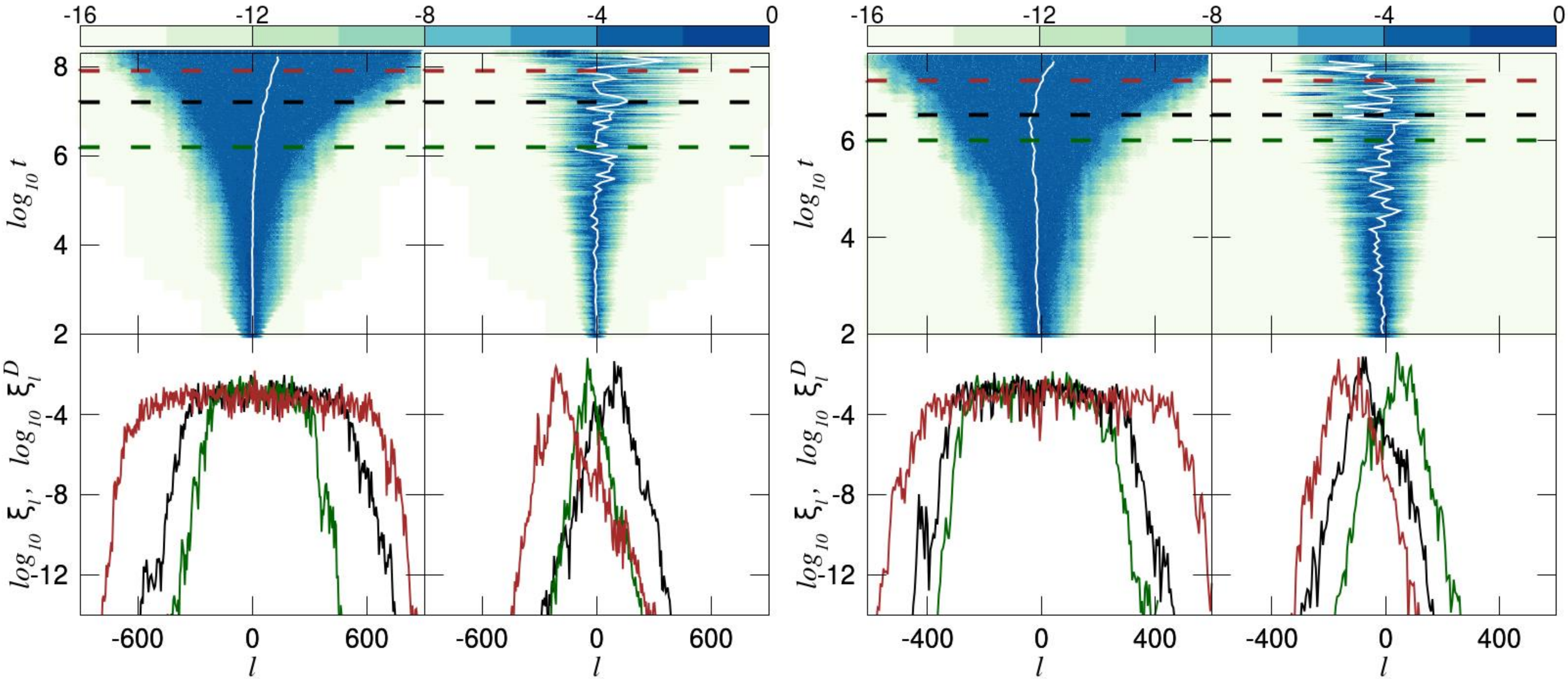
DDNLS: strong chaos W=3.5, L=21, β=0.72

Norm **DVD**



Strong Chaos: DKG and DDNLS

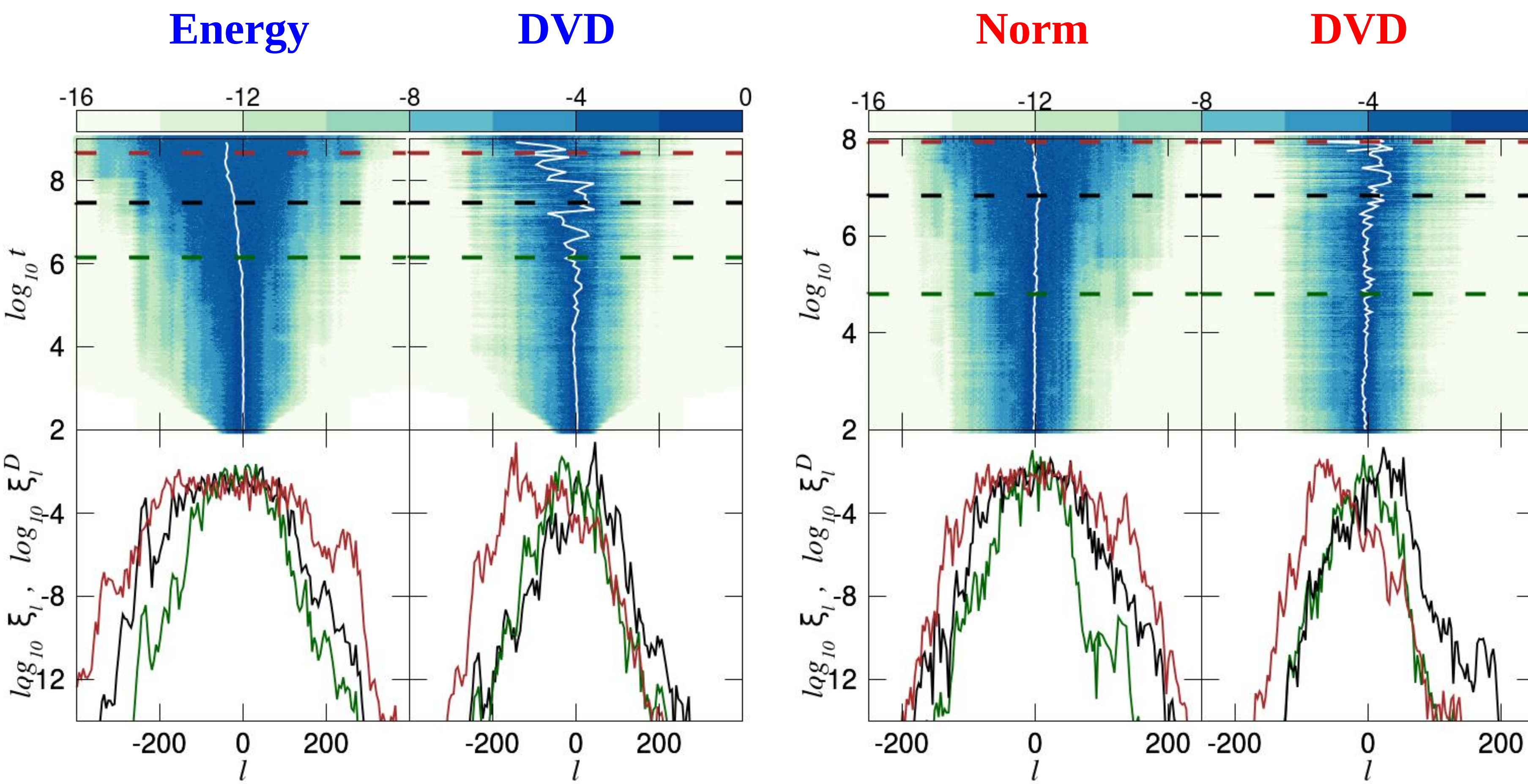
Energy DVD Norm DVD



DKG: W=3, L=83, E=8.3

DDNLS: W=3.5, L=21, β=0.72

Weak Chaos: DKG and DDNLS

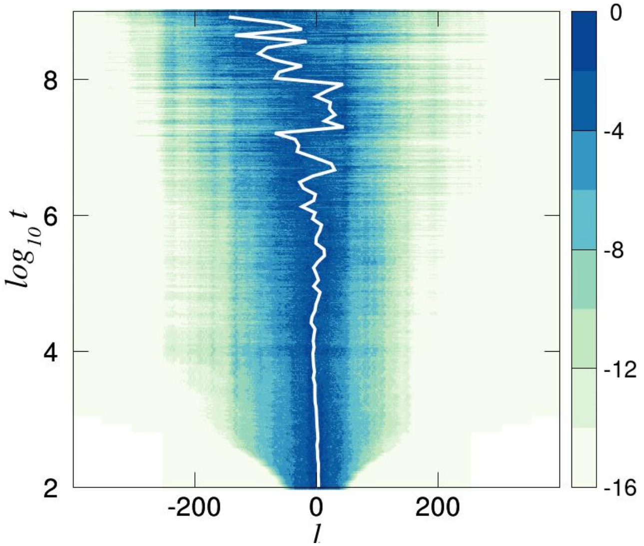


DKG: W=3, L=37, E=0.37

DDNLS: W=4, L=21, β=0.04

Characteristics of DVDs

KG weak chaos
L=37, E=0.37, W=3

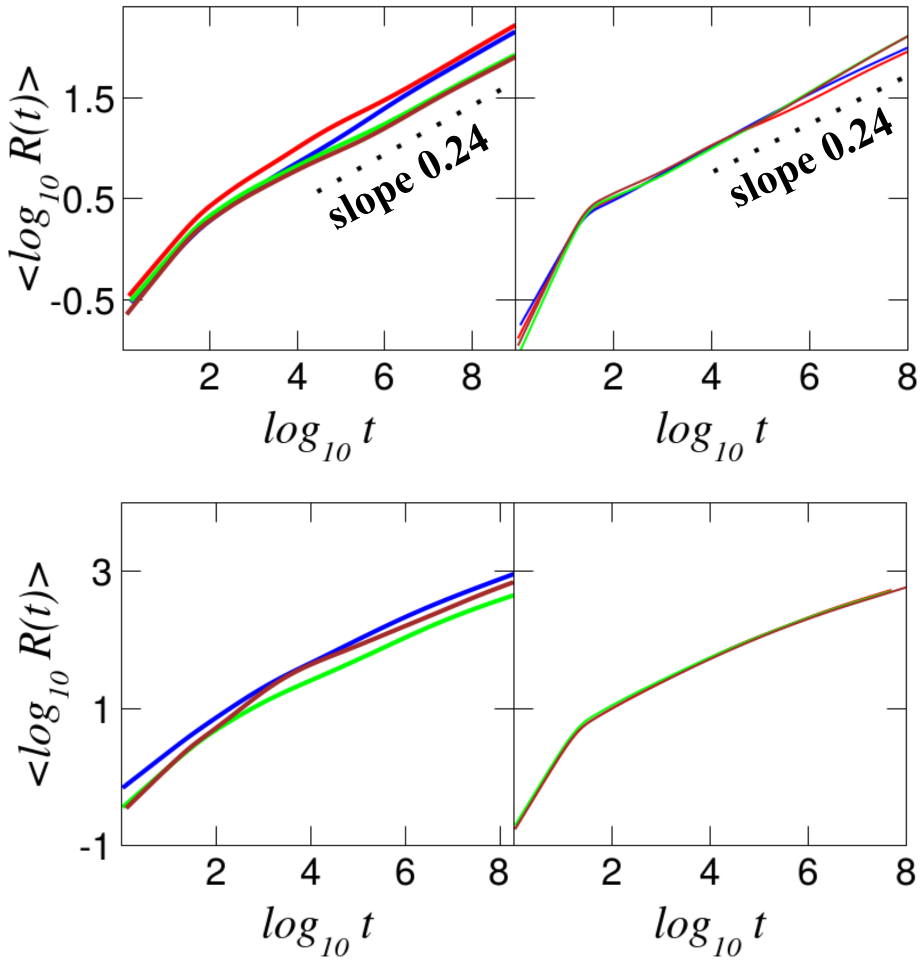


Range of the lattice visited by the DVD

$$R(t) = \max_{[0,t]} \left\{ \overline{l}_w(t) \right\} - \min_{[0,t]} \left\{ \overline{l}_w(t) \right\}$$

$$\overline{l}_w = \sum_{l=1}^{N} l \xi_l^D$$

DKG **DDNLS**



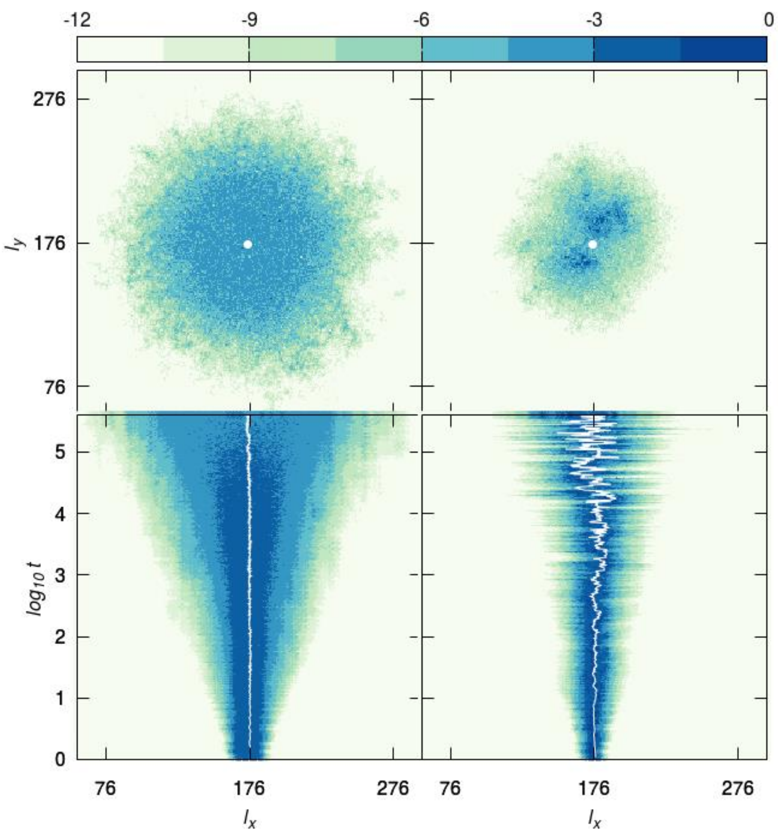
Weak chaos

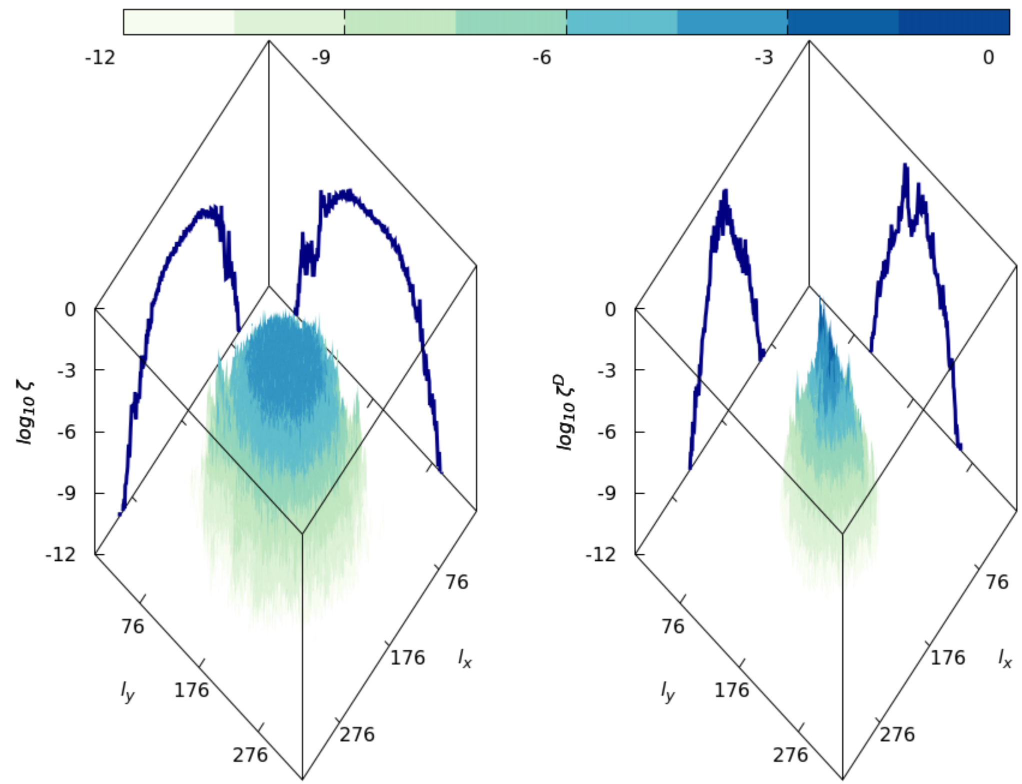
Strong chaos

Two-dimensional systems

DDNLS in 2 spatial dimensions (strong chaos)

[Many Manda, Senyange, & S., PRE (2020)]





Norm **DVD** **Norm** **DVD**

Two-dimensional systems

For more information on the chaotic dynamics of 2D disordered lattices attend the oral presentation of Bertin Many Manda

OC17 (Thursday 1 July at 16:00):

Nonequilibrium chaos of wave spreading in two-dimensional disordered lattices

Summary

- Both the DKG and the DDNLS models show similar chaotic behaviors
- The mLCE and the DVDs show different behaviors for the weak and the strong chaos regimes.
- Lyapunov exponent computations show that:
  - ✓ Chaos not only exists, but also persists.
  - ✓ Slowing down of chaos does not cross over to regular dynamics.
  - ✓ Weak chaos: mLCE ~ $t^{-0.25}$ - Strong chaos: mLCE ~ $t^{-0.3}$
- The behavior of DVDs can provide information about the chaoticity of a dynamical system.
  - ✓ Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.

B. Senyange, B. Many Manda & Ch. S.: 'Characteristics of chaos evolution in one-dimensional disordered nonlinear lattices', Phys. Rev. E, 98, 052229 (2018)

B. Many Manda, B. Senyange & Ch. S.: 'Chaotic wave packet spreading in two-dimensional disordered nonlinear lattices ', Phys. Rev. E, 101, 032206 (2020)

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Dynamics of disordered lattices

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Numerical integration of multidimensional Hamiltonian systems

Chaotic dynamics of nonlinear Hamiltonian systems

Oral presentation of Malcolm Hillebrand

OC17 (Thursday 1 July at 16:30):

Chaotic Dynamics in a Planar Model of Graphene

Poster presentation of Henok Tenaw Moges

Poster 15:

On the behavior of the Generalized Alignment Index (GALI) method for regular motion in multidimensional Hamiltonian systems

Henok Tenaw Moges[1], Thanos Manos[2], Charalampos Skokos[1]